Exam

Statistical Physics

Monday February 23, 2015
18:30-21:30

Read these instructions carefully before making the exam!

- **Write your name and student number on *every* sheet.**
- ***Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing*.**
- ***Language*; your answers have to be in English.**
- **Use a *separate* sheet for each problem.**
- **Use of a (graphing) calculator is allowed.**
- **This exam consists of 4 problems.**
- **The weight of the problems is Problem 1 (P1=25 pts); Problem 2 (P2=20 pts); Problem 3 (P3=20 pts); Problem 4 (P4=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.**
- **The grade of the exam is calculated as (P1+P2+P3+P4+10)/10.**
- **For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.***

PROBLEM 1

Score: a+b+c+d+e =5+5+5+5+5=25

Consider a solid that consists of a large number (N) of atoms with spin $\frac{1}{2}$, each of which has a fixed position in space. Each atom has a magnetic moment μ that can be aligned either parallel or anti-parallel with an external magnetic field B. We assume that the magnetic moment of one atom interacts only weakly with those around it. The solid is in equilibrium at temperature T.

The energy levels of a single atom are:

$\varepsilon_1 = -\mu B$: if the spin is parallel to the magnetic field B

$\varepsilon_2 = \mu B$: if the spin is antiparallel to the magnetic field B

We define the variable, $x = \frac{\mu B}{kT}$.

a) Give the single atom partition function Z_1 (express your answer in terms of x).

b) Give expressions for the population numbers n_1 and n_2 of the energy levels ε_1 and ε_2, respectively (express your answers in terms of x). Determine the values of n_1 and n_2 in the limit $T \to 0$ and in the limit $T \to \infty$.

c) Proof that the total energy E due to the spins of the N atoms is given by,

$$E = -NkTx\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d) Proof that the Helmholtz free energy F and the entropy S of the spins of the N atoms are given by,

$$F = -NkT\ln[e^x + e^{-x}]$$

$$S = Nk\left(\ln[e^x + e^{-x}] - x\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

e) Suppose the solid is in equilibrium at a temperature $T_1 = 4$ K. The magnetic field is then reduced from its initial value B_1 to a lower value $B_2 = 0.05B_1$ in a *reversible adiabatic* process. Calculate the temperature T_2 of the system after this process.

PROBLEM 2

Score: a+b+c+d =5+5+4+6=20

Consider a classical two-dimensional monoatomic gas that consists of N atoms with mass m. The ‘volume’ of this gas is given by a square with side L (and surface area $A = L^2$). In this problem you only have to consider the translational motion of the atoms.

a) Show that for an atom confined to this surface A, the number of states in which the atom has a momentum with its magnitude between p between p and $p + dp$ is given by:

$$f(p)dp = \frac{2\pi A}{h^2} pdp$$

b) Show that the single atom partition function is given by,

$$Z_1 = \frac{A2\pi mkT}{h^2}$$

c) Give an expression for the N-atom partition function and use this to show that the Helmholtz free energy F of the 2-dimensional gas is,

$$F = -NkT \left[\ln\left(\frac{A2\pi mkT}{Nh^2}\right) + 1 \right]$$

d) Calculate the surface tension τ of the 2-dimensional gas by using the expression of the Helmholtz free energy and the fundamental thermodynamic equation in two dimensions ($dE = TdS - \tau dA$).

PROBLEM 3

Score: a+b+c+d =5+5+5+5=20

According to quantum theory, the energy levels of a harmonic oscillator are given by $\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right)$; $n = 0,1,2 \cdots$, with ω the angular frequency of the oscillator. Assume that this oscillator is in equilibrium with a heat bath at temperature T.

a) Show that the partition function of the oscillator is given by,

$$Z_1 = \frac{1}{2\sinh\frac{x}{2}} \quad \text{with} \quad x = \frac{\hbar\omega}{kT}$$

b) Proof that the mean energy $\bar{\varepsilon}$ of the oscillator is given by:

$$\bar{\varepsilon} = \frac{1}{2}\hbar\omega\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}$$

Now consider an 1-dimensional solid that consists of N atoms. Assume that the solid can be described as a system of N independent oscillators (thus, use Einstein's theory!).

c) Show that the heat capacity C_V of this 1-dimensional solid can be written as:

$$C_V = Nk\left(\frac{x}{2}\right)^2\left(\frac{1}{\sinh^2\frac{x}{2}}\right)$$

Classically the Hamiltonian H of the 1-dimensional oscillator is,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

with m the mass of the oscillating atom and p and q the momentum and the position coordinate of the atom.

d) Show that the heat capacity that is predicted from this classical Hamiltonian and the equipartition theorem agrees with the high temperature limit of the heat capacity that follows from the result of subproblem c).

PROBLEM 4

Score: a+b+c+d+e=5+5+5+6+4=25

Consider a (three dimensional) ideal gas of ultra-relativistic electrons that is confined to a volume V. For an ultra-relativistic electron the contribution of its rest mass to the total electron energy E_e is negligible with respect to the contribution due to its momentum.

$$E_e^2 = p^2c^2 + m^2c^4 \approx p^2c^2 \Rightarrow E_e = pc$$

a) Show that for this gas of ultra-relativistic electrons the total number of particles is,

$$N = \frac{8\pi V}{h^3c^3}\int_0^\infty \frac{E_e^2 dE_e}{e^{\beta(E_e-\mu)}+1}$$

HINT: The density of states for a *spinless* particle confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$f(p)dp = \frac{V}{h^3}4\pi p^2 dp$$

b) Show that at zero absolute temperature ($T = 0$), the maximum energy E_F of an electron in this gas is,

$$E_F = hc\left(\frac{3}{8\pi}\frac{N}{V}\right)^{\frac{1}{3}}$$

c) Show that the total energy E of the ultra-relativistic electron gas at $T = 0$ is:

$$E = \frac{3}{4}NE_F$$

d) Calculate the pressure of the ultra-relativistic electron gas.

e) Explain how electrons at zero absolute temperature can have such high velocities that they are relativistic.

Solutions

PROBLEM 1

a)
Use the definition of the partition function: Eq. 2.23 (Mandl)

$$Z_1 = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} = e^{\beta\mu B} + e^{-\beta\mu B} = e^x + e^{-x}$$

b)
Probability that an atom is in state ε_1 is :

$$p_1 = \frac{e^{-\beta\varepsilon_1}}{Z_1} = \frac{e^x}{e^x + e^{-x}}$$

Probability that an atom is in state ε_2 is :

$$p_2 = \frac{e^{-\beta\varepsilon_2}}{Z_1} = \frac{e^{-x}}{e^x + e^{-x}}$$

There are N independent atoms (because weakly interacting), thus,

$$n_1 = Np_1 = N\frac{e^x}{e^x + e^{-x}} = N\frac{1}{1 + e^{-2x}}$$

$$n_2 = Np_2 = N\frac{e^{-x}}{e^x + e^{-x}} = N\frac{1}{1 + e^{2x}}$$

As a check we see that $n_1 + n_2 = N$

Limit $T \to 0$ then $\beta \to \infty$ and $x \to \infty$ and we have $n_1 \to N$ and $n_2 \to 0$, all spins are in the low energy state and are aligned with the magnetic field.

Limit $T \to \infty$ then $\beta \to 0$ and $x \to 0$ and we have $n_1 \to \frac{N}{2}$ and $n_2 \to \frac{N}{2}$, the spins are evenly distributed over both energy states.

c)

The mean energy of one atom is,

$$\bar{\varepsilon} = p_1\varepsilon_1 + p_2\varepsilon_2 = \frac{\varepsilon_1 e^x}{e^x + e^{-x}} + \frac{\varepsilon_2 e^{-x}}{e^x + e^{-x}} = \frac{-\mu B e^x + \mu B e^{-x}}{e^x + e^{-x}} = -kTx\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Thus for the N atoms,

$$E = N\bar{\varepsilon} = -NkTx\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d)

The Helmholtz free energy is given by,

$$F = -kT\ln Z_N = -kT\ln Z_1^N = -NkT\ln Z_1 = -NkT\ln[e^x + e^{-x}]$$

$$F = E - TS \Rightarrow S = \frac{E - F}{T} = -Nkx\frac{e^x - e^{-x}}{e^x + e^{-x}} + Nk\ln[e^x + e^{-x}] \Rightarrow$$

$$S = Nk\left(\ln[e^x + e^{-x}] - x\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

e)

In the reversible adiabatic process the entropy does not change, thus

$$S(T_1, B_1) = S(T_2, B_2)$$

This implies,

$$x_1 = x_2 \Rightarrow \frac{\mu B_1}{kT_1} = \frac{\mu B_2}{kT_2} \Rightarrow T_2 = \frac{B_2}{B_1}T_1 = 0.05 \times 4 = 0.2 \text{ K}$$

PROBLEM 2

a)

From the solution of the 2D-wave equation: $\varphi = A \sin k_x x \sin k_y y$ and taking this function to vanish at $x = y = 0$ and at $x = y = L$ results in,

$k_x = \frac{n_x \pi}{L}$ and $k_y = \frac{n_y \pi}{L}$ with n_x and n_y non-zero positive integers.

The total number of states with $|\vec{k}| < k$ is then given by, (the area of a quarter circle because we have only positive integers, with radius k divided by the area of the unit surface e.g. the surface of one state, in k-space).

$$\Gamma(k) = \frac{\frac{1}{4}\pi k^2}{\left(\frac{\pi}{L}\right)^2} = \frac{1}{4}\frac{L^2 k^2}{\pi}$$

The number of states between $k + dk$ and k is:

$$f(k)dk = \Gamma(k + dk) - \Gamma(k) = \frac{\partial \Gamma}{\partial k} dk = \frac{1}{2}\frac{L^2 k}{\pi} dk = \frac{1}{2}\frac{Ak}{\pi} dk$$

Converting to momentum $p = \hbar k = \frac{h}{2\pi} k$ we find,

$$f(p)dp = \left(\frac{2\pi}{h}\right)^2 \frac{1}{2}\frac{Ap}{\pi} dp = \frac{2\pi A}{h^2} p dp$$

b)

Single atom partition function (use substitution $p = x\sqrt{2mkT}$):

$$Z_1 = \int_0^\infty \frac{2\pi p A}{h^2} e^{-\beta\frac{p^2}{2m}} dp = \frac{2\pi A}{h^2} \int_0^\infty p e^{-\beta\frac{p^2}{2m}} dp = \frac{4mkT\pi A}{h^2} \int_0^\infty x e^{-x^2} dx = \frac{4mkT\pi A}{h^2}\frac{1}{2}$$
$$= \frac{A 2\pi mkT}{h^2}$$

c)

N-atom partition function:

$$Z_N = \frac{1}{N!}(Z_1)^N = \frac{1}{N!}\left(\frac{A2\pi mkT}{h^2}\right)^N$$

The Helmholtz free energy is:

$$F = -kT \ln Z_N = -kT \ln \frac{1}{N!} (Z_1)^N = -NkT \ln Z_1 - kT \ln N! \Rightarrow$$

$$F = -NkT \ln \left(\frac{A2\pi mkT}{h^2} \right) - NkT(\ln N - 1) = -NkT \left[\ln \left(\frac{A2\pi mkT}{Nh^2} \right) + 1 \right]$$

d)
From $F = E - TS \Rightarrow dF = dE - TdS - SdT$ and the fundamental equation for the 2-dimensional gas we find:

$$dF = -SdT - \tau dA$$

And thus,

$$\tau = -\left(\frac{\partial F}{\partial A} \right)_T = \frac{NkT}{A}$$

PROBLEM 3

a)

Use the definition of the partition function and $x = \frac{\hbar\omega}{kT} = \beta\hbar\omega$

$$Z_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\frac{x}{2}} \sum_{n=0}^{\infty} e^{-nx} = e^{-\frac{x}{2}} \frac{1}{1-e^{-x}} = \frac{1}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \frac{1}{2\sinh\frac{x}{2}}$$

b)

$$\bar{\varepsilon} = -\frac{\partial \ln Z_1}{\partial\beta} = -\left(\frac{\partial \ln Z_1}{\partial x}\right)\left(\frac{\partial x}{\partial\beta}\right) = -\left(\frac{\partial}{\partial x}\ln\left(\frac{1}{2\sinh\frac{x}{2}}\right)\right)(\hbar\omega) \Rightarrow$$

$$\bar{\varepsilon} = -2\sinh\frac{x}{2}\left(\frac{\partial}{\partial x}\frac{1}{2\sinh\frac{x}{2}}\right)(\hbar\omega) = -2\hbar\omega\sinh\frac{x}{2}\left(\frac{-1}{4\sinh^2\frac{x}{2}}\cosh\frac{x}{2}\right) = \frac{1}{2}\hbar\omega\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}$$

c)

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \left(\frac{\partial N\bar{\varepsilon}}{\partial T}\right)_V = N\left(\frac{\partial\bar{\varepsilon}}{\partial x}\right)_V\left(\frac{\partial x}{\partial T}\right)_V = \frac{1}{2}N\hbar\omega\left(\frac{\partial}{\partial x}\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}\right)_V\left(-\frac{\hbar\omega}{kT^2}\right) \Rightarrow$$

$$C_V = -\frac{1}{2}Nkx^2\left(\frac{\partial}{\partial x}\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}\right)_V = -\frac{1}{2}Nkx^2\left(\frac{-\frac{1}{2}}{\sinh^2\frac{x}{2}}\right) = Nk\left(\frac{x}{2}\right)^2\left(\frac{1}{\sinh^2\frac{x}{2}}\right)$$

Consider a completely degenerate perfect gas of N fermions confined to a volume V at zero absolute temperature.

d)

The equipartition theorem states that each quadratic term in the Hamiltonian results in a contribution of $\frac{1}{2}kT$ to the energy of the system. In the classical Hamiltonian we find two quadratic terms thus a contribution of $\frac{1}{2}kT + \frac{1}{2}kT = kT$ to the energy. The total energy of N oscillators is: NkT, which leads to a heat capacity of Nk.

For high temperatures we have $x \to 0$. In this limit we find:

$$\lim_{x\to 0} C_V = \lim_{x\to 0} Nk\left(\frac{x}{2}\right)^2\left(\frac{1}{\sinh^2\frac{x}{2}}\right) = \lim_{x\to 0} \frac{Nk\left(\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2} = Nk$$

PROBLEM 4

a)

Use the hint to find the density of states as a function of energy. Remember to multiply with a factor of 2 because of the two spins states of the electron.

$$f(p)dp = \frac{2V}{h^3} 4\pi p^2 dp = \frac{2V}{h^3} 4\pi \left(\frac{E_e}{c}\right)^2 \frac{dE_e}{c} = \frac{8\pi V}{h^3 c^3} E_e^2 dE_e = f(E_e) dE_e$$

Total number of electron is given by,

$$N = \int_0^\infty n(E_e) f(E_e) dE_e$$

with,

$$n(E_e) = \frac{1}{e^{\beta(E_e - \mu)} + 1}$$

the Fermi-Dirac distribution.

Thus,

$$N = \frac{8\pi V}{h^3 c^3} \int_0^\infty \frac{E_e^2 dE_e}{e^{\beta(E_e - \mu)} + 1}$$

b)

At absolute zero the electron gas is completely degenerate and all energy levels up to a maximum level E_F (Fermi level) are filled with 1 electron each and all the other energy levels are empty. Thus, $n(E_e) = 1$ if $E_e < E_F$ and $n(E_e) = 0$ if $E_e > E_F$. Consequently, the total number of electrons N is given by,

$$N = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E_e^2 dE_e = \frac{8\pi V}{3h^3 c^3} E_F^3 \Rightarrow E_F = hc \left(\frac{3}{8\pi} \frac{N}{V}\right)^{\frac{1}{3}}$$

c)

$$E = \int_0^\infty E_e n(E_e) f(E_e) dE_e = \int_0^{E_F} E_e f(E_e) dE_e \Rightarrow$$

$$E = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E_e^3 \, dE_e = \frac{8\pi V}{4h^3 c^3} E_F^4 = \frac{3}{4} N E_F$$

d)

$$dE = TdS - PdV \underset{T\to 0}{\Longrightarrow} P = -\frac{dE}{dV}$$

$$P = -\frac{d}{dV}\left(\frac{3}{4} N E_F\right) = -\frac{d}{dV}\left(\frac{3}{4} Nhc \left(\frac{3}{8\pi}\frac{N}{V}\right)^{\frac{1}{3}}\right) = \frac{1}{4} hc \left(\frac{3}{8\pi}\right)^{\frac{1}{3}} \left(\frac{N}{V}\right)^{\frac{4}{3}}$$

e)

Because of the Pauli exclusion principle electrons (fermions) cannot all be in the ground state but occupy different energy levels up to the Fermi energy E_F. When the system is compressed to high density the ratio N/V increases and thus the Fermi energy E_F increases. This implies that also the momenta of the electrons increase possibly up to (ultra) relativistic values. This feature originates from the uncertainty relation: $\Delta p \Delta x \geq \hbar$. When the electron gas is compressed to high density the distance Δx between the electrons decreases, and the momenta Δp of the electrons increase.